REDUCING STORMWATER RUNOFF: A MATHEMATICAL MODEL OF GREEN ROOF DESIGN

Abstract

Green roofs are one of the most efficient ways to decrease stormwater runoff. Optimizing green roof design helps maximize stormwater infiltration. This paper proposes a water balance model of green roofs and validates it with data measured on the green roof at Milwaukee.

The water balance model proposed in this paper can be programmed using any simulation software, which explains the effects of vegetation coverage, soil porosity and depth, and plant type on stormwater runoff. Stormwater runoff can be reduced by more vegetation coverage, greater soil porosity and depth, and plants with lower internal leaf resistance and larger leaf size.

Authors

Jing Hong and Dennis Michael Utzinger
University of Wisconsin

Keywords

Green roof, stormwater, runoff, simulation

Introduction

Green roofs help to reduce stormwater runoff. They manage precipitation and other moisture in two ways: absorption and evapotranspiration. In turn, less water runoff lowers the risk of flooding and water contamination. The goal of this research is to create a water balance model of a green roof to estimate stormwater runoff. The proposed model was validated by data gathered from the green roof of Golda Meir Library at Milwaukee. Using the proposed water balance model, the relevant characteristics of a green roof can be adjusted to minimize the stormwater runoff.

Water Balance Model

In general, the component terms of the water balance equation of soil are shown in the following equation (Oke, 2002):

$$p = \Delta S + E \times \Delta t + \Delta r$$

p	Precipitation, kg/m²
ΔS	Soil moisture change, kg/m²
E	Evapotranspiration, kg/(m²·s)
Δt	Time period, s
Δr	Net runoff, kg/m²

The precipitation, *p*, which is also the amount of rainfall in the summer, is measured by the weather station located on the east roof. Soil moisture change can be calculated by the water content measured by the weather station equipped with five moisture sensors installed on the green roofs of Golda Meir Library at Milwaukee.

$$\Delta S = (w_{i+1} - w_i) \times d \times \frac{1000kg}{1\,m^3}$$

ΔS	Soil moisture change, kg/(m²·s)
w_{i+1}	The water content at the end of the time step, m³/m³
w_i	The water content at the beginning of the time step, m³/m³
d	Soil depth, m

The precipitation monitored by the weather station is the rainfall depth accumulated in 15 minutes. Therefore, instantaneous evaporation rate should be multiplied by the number of time periods.

$$w_{i+1} = \frac{p - E \times \Delta t - \Delta r}{d \times 1000} + w_i$$

The evaporation rate, *E*, of uncovered soil is different from the soil beneath canopy.

Regarding to the evaporation rate discussed in energy balance, the evaporation rate for the soil without vegetation coverage is (Oke, 2002):

$$E_{soil} = \rho_{air} C V \Delta\bar{q}$$

E_{soil}	The rate of evaporation from the soil, kg/(m²s)
ρ_{air}	Air density, kg/m³
C	Dalton number, approximately 1.5 × 10^{-3}
V	Mean wind speed, m/s
$\Delta\underline{q}$	The difference in humidity between the surface and the air, kg/kg

The humidity can also be calculated with a known temperature and pressure (Gates, 2012):

$$q = \frac{0.622e}{P - 0.379e} \cong \frac{0.622e}{P}$$

e	Water vapor pressure Pa
P	Total atmospheric pressure Pa

Tetens' formula for temperatures above 0°C define the water vapor pressure as indicated below (Monteith & Unsworth, 2013):

$$e = 0.61078\exp\left(\frac{17.27T}{T + 237.3}\right)$$

T	Air temperature, °C
e	Water vapor pressure kPa

For the bare soil,

$$\Delta\bar{q} = \frac{0.622}{P} 0.61078(\exp\left(\frac{17.27T_{soil}}{T_{soil} + 237.3}\right) - \exp\left(\frac{17.27T_{air}}{T_{air} + 237.3}\right)$$

T_{soil}	Soil surface temperature, °C
T_{air}	Ambient temperature, °C

Therefore,

$$E_{soil} = \rho_{air} C V \cdot \frac{0.622}{P} 0.61078(\exp\left(\frac{17.27T_{soil}}{T_{soil} + 237.3}\right) - \exp\left(\frac{17.27T_{air}}{T_{air} + 237.3}\right)$$

For the soil underneath the canopy,

$$E_{veg} = \frac{\rho_a \cdot \frac{BP_{w,l}}{(P_{total} - P_{w,l})} - h \cdot \rho_a \cdot \frac{BP_{w,a}}{(P_{total} - P_{w,a})}}{r_l - r_a}$$

ρ_a	Dry air density, kg/m³
B	0.622 for air, kg/kg
$P_{w,l}$	Water vapor pressure on leaf, mbar
$P_{w,a}$	Water vapor pressure of air, mbar
P_{total}	Ambient total pressure, mbar
r_l	Internal leaf resistance, 100-2000 s/m
r_a	A surface boundary-layer resistance

The vegetation coverage of the roof is represented by *LAI* in the following equation. The comprehensive evaporation rate can be estimated as:

$$E = E_{veg} + E_{soil} = LAI \cdot \frac{\rho_a \cdot \frac{BP_{w,l}}{(P_{total} - P_{w,l})} - h \cdot \rho_a \cdot \frac{BP_{w,a}}{(P_{total} - P_{w,a})}}{r_l - r_a} + (1 - LAI) \cdot \rho_{air} C V \cdot \frac{0.622}{P} 0.61078(\exp\left(\frac{17.27T_{soil}}{T_{soil} + 237.3}\right) - \exp\left(\frac{17.27T_{air}}{T_{air} + 237.3}\right)$$

LAI: Leaf Area Index

To estimate the maximum water content of the saturated soil, I conducted an experiment. In this experiment, 250 to 1000 ml water was added into 250 ml of soil to see how much water ran out of the soil. The average maximum water content of the soil was found to be 0.384 cm³/cm³. The calculation is shown in Equation 11.

$$WC_{max} = \frac{V_{water,sat} - V_{run-off}}{V_{soil}}$$

WC_{max}	Maximum water content of the soil sample, ml/ml
$V_{water,sat}$	The water volume of saturated soil sample, ml
$V_{run-off}$	The runoff water volume of saturated soil sample, ml
V_{soil}	The soil sample volume, ml

The measured maximum water content reading of the soil moisture sensor is 0.383 cm³/cm³. The accuracy of the soil moisture sensor is ±0.031 m³/m³. The error between the calculated and the measured is 0.001, which is acceptable. Therefore, when the reading of the soil moisture reaches 0.383, the water ratio in the soil has reached its maximum water content.

However, the experiment of studying the soil water absorption capacity was done in a measuring cup, for a green roof system with drainage composite beneath the growing medium, water running-off occurs before the soil gets saturated due to the gravity and pores in the soil. I conducted another experiment to study the water runoff ratio. This experiment mimicked a green roof drainage system with a strainer set underneath the soil. The experiment results revealed that when adding 100 ml water into the 250 ml soil sample, the soil water content became stable and the soil absorbing capacity declined.

Finding the ratio between the water runoff volume and added-in water volume can estimate the water runoff amount during a rain event, as shown in Figure 1.

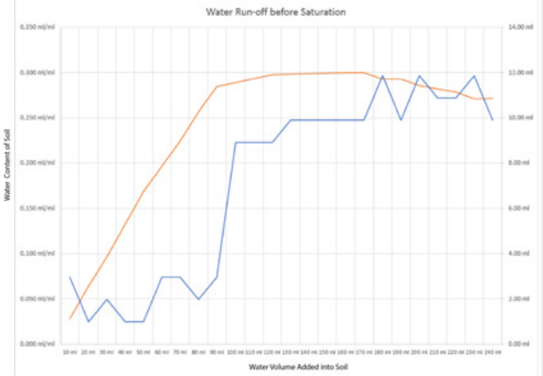

Figure 1: Water runoff ratio (Orange line: Water Content of Soil. Blue line: Run-off Water Volume).

Added-in Water	Runoff Water	Water Content	Runoff Ratio
10 ml	2.97 ml	0.028 ml/ml	0.30
10 ml	0.99 ml	0.064 ml/ml	0.10
10 ml	1.98 ml	0.096 ml/ml	0.20
10 ml	0.99 ml	0.132 ml/ml	0.10
10 ml	0.99 ml	0.168 ml/ml	0.10
10 ml	2.97 ml	0.196 ml/ml	0.30
10 ml	2.97 ml	0.225 ml/ml	0.30
10 ml	1.98 ml	0.257 ml/ml	0.20
10 ml	2.97 ml	0.285 ml/ml	0.30
10 ml	8.91 ml	0.289 ml/ml	0.89
10 ml	8.91 ml	0.294 ml/ml	0.89
10 ml	8.91 ml	0.298 ml/ml	0.89
10 ml	9.90 ml	0.298 ml/ml	0.99
10 ml	9.90 ml	0.299 ml/ml	0.99
10 ml	9.90 ml	0.299 ml/ml	0.99
10 ml	9.90 ml	0.300 ml/ml	0.99
10 ml	9.90 ml	0.300 ml/ml	0.99
10 ml	11.87 ml	0.293 ml/ml	1.19
10 ml	9.90 ml	0.293 ml/ml	0.99
10 ml	11.87 ml	0.285 ml/ml	1.19
10 ml	10.88 ml	0.282 ml/ml	1.09
10 ml	10.88 ml	0.278 ml/ml	1.09
10 ml	11.87 ml	0.271 ml/ml	1.19
10 ml	9.90 ml	0.271 ml/ml	0.99

Table 1: Water runoff ratio estimation.

Table 1 provides us four pieces of information:

- When the water content is lower than 0.028, there will not be water runoff. When the first 10ml water was added into the 250 ml soil, there was only 2.97 ml water ran off the soil. That meant 7.03 ml water was completely absorbed by the soil. So, when the water content is lower than 7.03/250 ml/ml. This means that there is no water runoff.
- When the water content is lower than 0.196, the water runoff ratio is about 0.1 of the added-in water.
- When the water content is greater than 0.196, but lower than 0.285, the water runoff ratio is about 0.3 of the added-in water.
- When the water content is larger than 0.289, but lower than the maximum water content of the soil, the water runoff ratio is about 0.89 of the added-in water.

Cooperate the information harvested in Table 1 to estimate the Δr_i:

$$\Delta r_i = \begin{cases} 0, & p_i = 0 \\ 0, & w_{i-1} < 0.028 \\ p_{i+1}, & \sum_{i=1}^{n} p_i \geq (0.383 - w_{i-1})\, d \\ 0.89 p_i, & \sum_{i=1}^{n} p_i \geq (0.30 - w_{i-1})\, d \\ 0.3 p_i, & \sum_{i=1}^{n} p_i \geq (0.20 - w_{i-1})\, d \\ 0.1 p_i, & \sum_{i=1}^{n} p_i < (0.20 - w_{i-1})\, d \end{cases}$$

w	Instantaneous water content, m^3/m^3
d	Soil depth, mm
p	Precipitation, mm
n	The number of time step of the accumulated rainfall

The number of time step of the accumulated rainfall, n, is determined by the time the saturated soil can be completely dry. By observing the water content variation in the dry season and solving Equation 12 to get w_{i+1} with different n, I found that six days of accumulated rain was the best estimation of n. Figure 2 shows how the modeled water content fits the measured water content with different n settings.

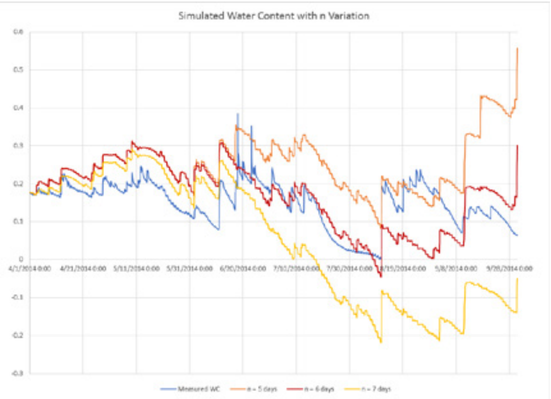

Figure 2: Testing variable n with 5, 6, and 7 days.

Model Validation

The mathematical models of green roofs' mass transfer are simulated using the numerical computing software MatLab. The coefficients in the models were based on experiments, estimation, and known measured results. The modeled water contents for the water balance model were compared with the measured ones. Equation 3 was simulated in MatLab to calculate the water content in each time step and then validated by the measured data in the same time step. The root-mean-square deviation (RMSD) and the standard error (SE) of the differences was used as the criterion to investigate the errors. The same calculation is applied to the mass balance model. The average difference of the simulation is 0.0197 m^3/m^3. The RMSD is 0.0769 m^3/m^3 and the SE is 0.000561. The comparison between the measured and modeled water content is shown in Figure 3.

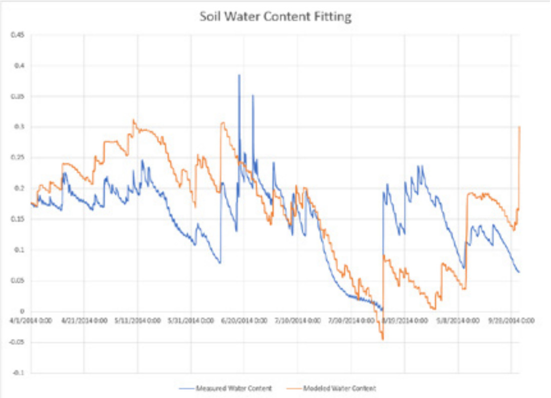

Figure 3: Measured and modeled soil water content comparison from April to September.

Application of the Water Balance Model

SOIL WATER CONTENT

Between June 20 and August 10 there was a dry season with minimal rain. After the dry season, there were a few heavy rainfalls between August 11 and August 25.

Figure 4 shows the big storms between June 18 and June 19. On June 17, the day before the rain came, the soil moisture change, Δ*S*, was negative. The soil lost water because of evaporation. On June 18, when there was a rainfall of 28 mm, the runoff was about 23 mm, and the water content change was about 10 mm. The soil absorbed about 18% of the rain. However on June 19, the day there was a storm with about 65 mm rainfall, 100% of the rain ran off the roof and the soil absorbed nearly zero water and the plant evaporation was also minimal. On June 20, there was approximately 14 mm rainfall, which was only half of the rainfall of June 18, but 100% of the rain ran off the roof, and the soil absorbed minimal water. The same situation occurred on June 21. Even with 4 mm of rainfall, the soil still failed to absorb any water, and 100% of the rainfall left the soil.

As shown in Figure 5, there was a dry period in Milwaukee from July 21 to August 10, 2014. There were only four days of minimal rain, and the green roof completely absorbed the rain without any runoff. The soil moisture change was negative throughout this period, which means the water either evaporated through the soil or the plants. Since Δ*S* were getting smaller during this period, I believe the total water amount in the soil was decreasing at this time.

After a 3-week dry period, the soil became porous, which increases its ability to retain water. Therefore, the 3 mm rainfall on August 11 was completely absorbed, and half of the 22.5 mm rainfall on August 12 was absorbed. In those cases, a total of 25.5 mm rain fell on the green roof, but only 11.25 mm ran off the roof. However, on August 13, when the water content in the soil had reached its maximum water absorption, 100% of the rainfall ran off the roof.

As shown in Figure 6, after three days of rain from August 11 to 13, it did not rain again until August 18. Unlike the rain on August 11, on August 18 the rain was not completely absorbed by the green roof and had approximately 50% runoff. August 11 and August 18 were both the first day of rain after a dry period, but the green roof reacted to the rain on these two days in different ways. The main reason was that the soil was dry enough to absorb all the rain on August 18, which means the water absorbed by the soil between August 11 to August 13 still affected the water absorption on August 18. This assumption could be proved by the sum of Δ*S*. The sum of the positive Δ*S* between August 11 to 13 was larger than the sum of the negative Δ*S* between August 13 and 17, which means the water restored between August 11 and 13 was not completely drained or evaporated. The moist soil had lower capacity to absorb incoming rain.

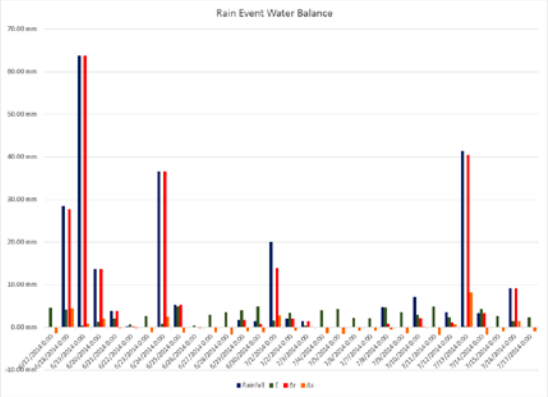

Figure 4: Water balance between June 17 and July 17, 2014.

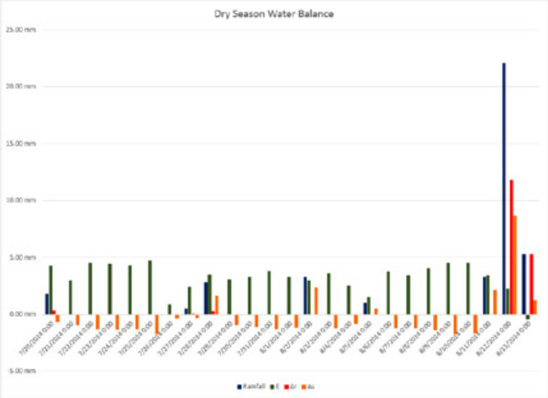

Figure 5: Water balance between July 20 and August 13, 2014.

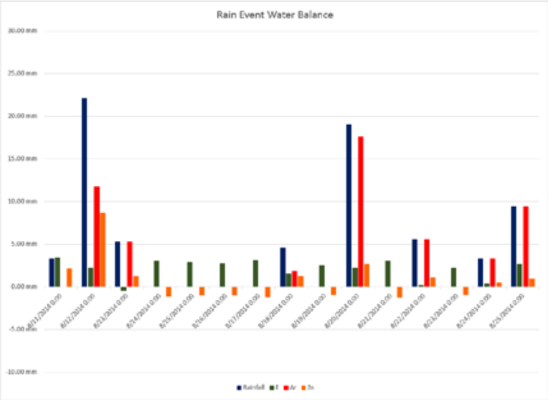

Figure 6: Water balance between August 11 and August 25, 2014.

THE EFFECTS OF VEGETATION COVERAGE, SOIL DEPTH, AND PLANT TYPE ON STORMWATER RUNOFF

The goal of a green roof is to reduce stormwater runoff and retain water in the soil. The precipitation, p, is not controllable. But the vegetation coverage LAI, soil depth, d, plants' internal leaf resistance, r_i, and leaf dimension, $W^{0.2}D^{0.3}$, can be planned in the schematic design phase.

$$p = E \times \Delta t + \Delta r + \Delta S = E \times \Delta t + \Delta r + (w_{i+1} - w_i) \times d$$

Based on Equation 13, when the rainfall is a constant, if E is increased, the Δr declines. However, regarding previous analysis, the evaporation rate, E, does not directly reduce the stormwater runoff, but it can reduce the water contained in the soil. This will make the soil have more space to retain an upcoming rainfall. Therefore, the larger the evaporation, the higher potential to reduce runoff. Three elements, vegetation coverage, and internal leaf resistance and dimension, have an impact on evaporation. To examine the effects of these three elements on evaporation, I assumed some comparative variables for them.

As shown in Equation 8, the evaporation consists of two parts: evaporation at the soil surface and the evapotranspiration through the plants. The vegetation coverage LAI is the percentage of the vegetation on the green roof; it determines the total evaporation rate of a green roof. To examine the effect of LAI on E_{total}, I simulated the E_{total} with different LAI settings. Figures 7 and 8 show the evaporation rates with LAI = 0.5, 0.6 and 0.7. The result shows that the evaporation rates increase along with the increasing LAI. The main reason is that the plants have higher evaporation rates than the soil surface. This conclusion was also proved in the energy flux density analysis.

Figures 7 and 8 show the effect of vegetation coverage on the evaporation rate during a rainy season and a dry season. Vegetation coverage does not impact the evaporation rate during a rain period as much as during the dry period. That means if a designer increases the vegetation coverage, the evaporation rates during the dry period will be increased. That will shorten the time to dry the soil and retain more water when it rains again.

Figures 9, 10, and 11 show the comparison of evaporation rates of the plants with different internal leaf resistances and boundary-layer resistances (leaf size). The comparison shows that the evaporation of the native plants is the largest among the tested plants, and that grass has the second-largest. Sedums have the least evaporation.

The soil porosity and depth also have an impact on the water runoff. More porous soil absorbs more water proportionally. Similarly, deeper soil has the capacity to absorb a correspondingly greater amount of water until saturation is reached.

To examine these effects, I simulated the water runoff with porosity and soil depth both 50% higher than in the original setting.

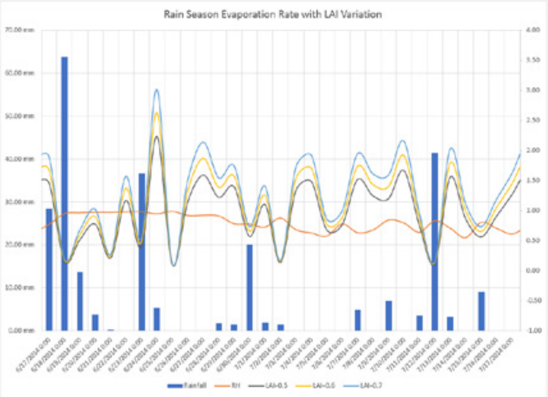

Figure 7: Evaporation rates with LAI = 0.5, 0.6 and 0.7 between June 17 and July 17, 2014.

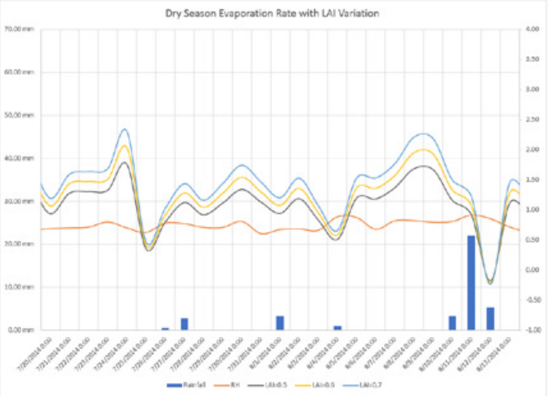

Figure 8: Evaporation rates with LAI = 0.5, 0.6 and 0.7 between July 20 and August 13, 2014.

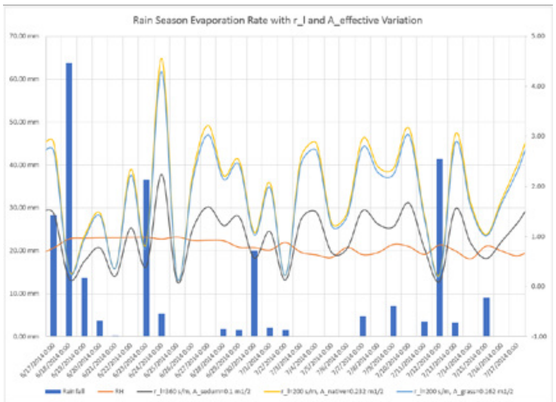

Figure 9: Evaporation rates with internal leaf resistance and leaf size variation between June 17 and July 17, 2014.

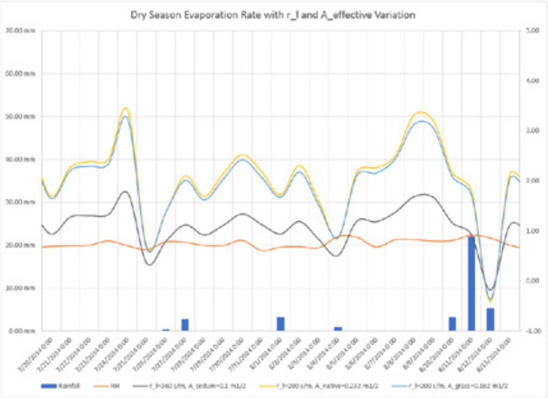

Figure 10: Evaporation rates with internal leaf resistance and leaf size variation between July 20 and August 13, 2014.

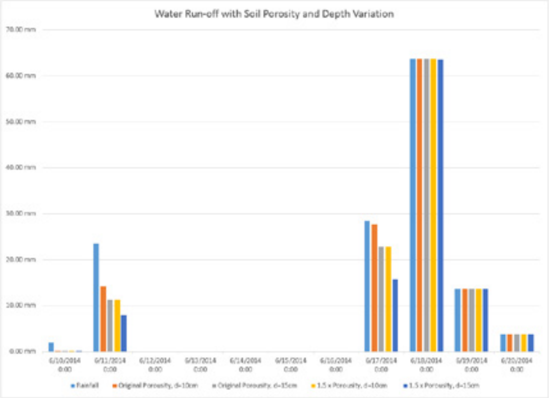

Figure 11: Water runoff comparison with different soil porosity and depth between June 10 and June 20, 2014.

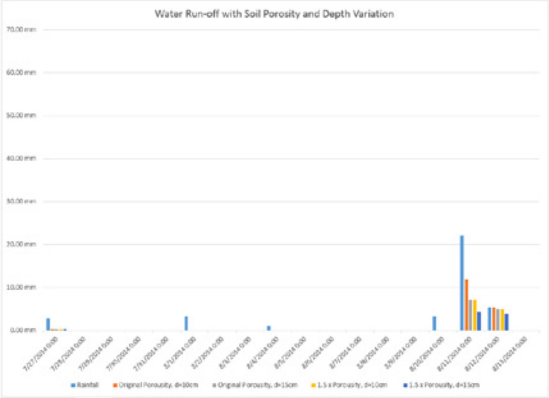

Figure 12: Water runoff comparison with different soil porosity and depth between July 27 and August 13, 2014.

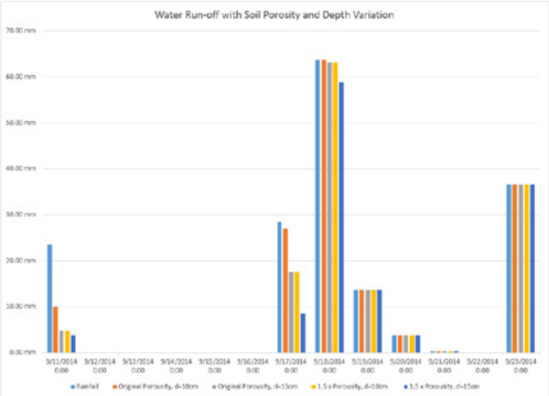

Figure 13: Water runoff comparison with different soil porosity and depth between September 11 and September 23, 2014.

Figure 11 shows the water runoff comparison after a 5-day dry period. On the first day of rain, the original porosity, original 10 cm-deep soil, had little water absorption and high runoff. The original porosity, 1.5 × depth soil and 1.5 × porosity, original depth soil had similar intermediate level of water runoff. The 1.5 × porosity, 1.5 × depth soil had the least runoff, at about 60% of the rainfall. When it continuously rained, all the soil types became saturated and had the same 100% runoff.

Figure 12 shows the water runoff comparison after a long dry period. Between July 27 and August 10, rainfall never exceeded a depth of 5 mm, and was completely absorbed by all four types of soil. On August 11, the first day it rained more than 20 mm, the original soil had approximately 50% runoff, the original porosity, 1.5 × depth soil and 1.5 × porosity, original depth soil had approximately 32% runoff, and the 1.5 × porosity, 1.5 × depth soil had approximately 20% runoff. On August 12, the second day it rained around 5 mm, the 1.5 × porosity, 1.5 depth soil showed 20% less water runoff than the other three types of soil. The other three types of soil had similar water runoff amounts, which were close to the rainfall amount.

Figure 13 shows another runoff comparison after a five-day dry period. The rain water runoff situation during this time was very similar to that of June 17 to June 20. On the first day of a series of rain showers, the 1.5 × porosity, 1.5 × depth soil absorbed the most rain; the original porosity, 1.5 × depth soil and 1.5 × porosity, original depth soil similarly reduced water runoff; and the original soil type had the most water runoff. When rain continued, and all the soil types reached their absorption capacity, all the rain ran off the roof regardless of soil porosity and depth. Results during the spring, when there were multiple days with small amounts of rain, were similar.

Conclusion

Four elements, vegetation coverage, soil depth, soil porosity, and plant type, assist in reducing stormwater runoff.

Increasing the vegetation coverage and minimizing the bare soil areas helps increase the total evaporation rate of a green roof, decreases the water content of the soil, and hence increases the soil's capacity to absorb and evaporate incoming stormwater.

The water evaporated by vegetation reduces the soil water content ratio, which increases the soil's absorption capacity. To increase the evaporation rate, a plant with low internal leaf resistance and large leaf size is recommended. The native plant performs best in increasing the evaporation rate. The grass is the second best and the sedum is the worst.

Increasing the soil porosity also increases the soil absorption. This means that selecting a high-absorption soil type can efficiently reduce water runoff. Some companies even developed artificial green roof growing media to improve upon the porosity of the best performing real soil.

Increasing soil depth is a way to increase the volume of a rain "container." However, in reality, an oversized soil layer for a green roof will increase the structural burden on a building. Therefore, increasing soil depth to reduce stormwater runoff is potentially problematic.

Acknowledgment

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